UCTP101.00

Supersymmetric Wilson Lines and Loops, and Super Non-Abelian Stokes Theorem

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Abstract

We generalize the standard product integral formalism to incorporate Grassmann valued matrices and show that the resulting supersymmetric product integrals provide a natural framework for describing supersymmetric Wilson lines and Wilson loops. We use this formalism to establish the supersymmetric version of the non-Abelian Stokes theorem.

1 Introduction

The notion of Wilson loop [1, 2] provides a systematic method of obtaining gauge invariant observables. Its standard applications range from particle phenomenology and lattice field theories to strings and topological gauge theories. More recently, in the context of the AdS/CFT correspondence [3], an interesting connection between Wilson loops in supersymmetric gauge theories and membranes in supergravity theories has been suggested [4]. In view of this and other important developments in supersymmetric gauge theories, it is natural to ask whether the notions of Wilson line and Wilson loop permit a supersymmetric generalization. Some formal work in this direction was carried out early in the development of supersymmetric gauge theories [5, 6, 7, 8, 9]. There are also some recent suggestions in the $N = 4$ case [10]. In contrast to these attempts, our aim is to construct supersymmetric Wilson lines and Wilson loops in terms of supersymmetric product integrals. For non-supersymmetric gauge theories, it has been shown recently [11] that standard product integrals [12] provide a natural framework for describing Wilson lines and Wilson loops. This is because they have a built-in feature for keeping track of the order of matrices in path ordered quantities. The main purpose of the present work is to extend these results to theories which involve supersymmetric matrices. Thus, our construction of supersymmetric Wilson lines and loops is the natural supersymmetric extension of the definition of their non-supersymmetric counter parts. This will permit us to give, among other things, an unambiguous proof of the supersymmetric version of the non-Abelian Stokes theorem.

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To provide a supersymmetric generalization of the notions of Wilson line and Wilson loop in terms of product integrals, we must address a number of questions. The first among these has to do with the fact that in supersymmetric gauge theories, the superfields have values in a Grassmann algebra. To be able to explore the properties of these theories in terms of product integrals, we must first ensure that Grassmann valued product integrals exist. We address this question in Section 2, where we construct supersymmetric product integrals and explore their properties. In Section 3, we use this formalism to define supersymmetric Wilson lines and Wilson loops. In Section 4, we construct a surface integral representation for the supersymmetric Wilson loop, thus establishing the supersymmetric version of the non-Abelian Stokes theorem. As a further confirmation of this theorem, in Section 5, we show the gauge covariance of the surface integral representation of the super Wilson loop operator. Section 6 is devoted to concluding remarks.

2 Supersymmetric Product Integrals

Comprehensive accounts of ordinary product integrals and their applications exist in the literature [11, 12]. Here we mention in passing that the justification for the word "product" lies in the property that the product integral is to the product what the ordinary integral is to the sum and that one of their most common applications is in solving systems of linear differential equations of the form

$$\mathbf{y}'(\mathbf{s}) = A(s)\,\mathbf{y}(\mathbf{x}), \quad \mathbf{y}(\mathbf{s_0}) = \mathbf{y_0}. \tag{1}$$

The solution of this system can be constructed in terms of the limit of the finite ordered product [12]: $\Pi_p(A) = \prod_{k=1}^{n} e^{A(s_k)\Delta s_k}$. In this expression, $\Delta s_k = s_k - s_{k-1}$ for $k = 1, ..., n$, where $\{s_0, s_1, ..., s_n\}$ is a partition of the real interval $[a, b]$. In the limit of large n and under suitable conditions, this ordered product leads to the definition of the product integral.

The properties of standard product integrals rest heavily on the Banach algebra structure of matrix valued functions [12]. In supersymmetric theories, the corresponding matrices take values in a Grassmann algebra. Since product integrals are products of the exponentials of the matrix valued functions, and in a supersymmetric theory the exponents must necessarily belong to the even part of the Grassmann algebra, we expect intuitively that all the properties of standard product integrals can be extended to supersymmetric product integrals. To put this on firm mathematical foundation, we must specify a suitable norm on the Grassmann algebra, with respect to which supersymmetric matrices also acquire a Banach algebra structure.

The Banach algebra structure of the Grassmann algebra is well known [13]. Consider for definiteness the finite dimensional Grassmann algebra generated by the anticommuting quantities $\theta^1, \theta^2, \ldots, \theta^p$. In this case, a generic element of the algebra can be written as a linear combination of the products $\theta^{i_1}\theta^{i_2}\ldots\theta^{i_k}$, $k = 0, \ldots, p$, with complex coefficients $a_{i_1 i_2 \ldots i_k}$. As a complex vector space the Grassmann algebra is 2^p dimensional. A norm on the above vector space (more precisely a valuation of the algebra) can be defined as the sum of the moduli of the coefficients. For example in the Grassmann algebra generated by a single element θ, the norm of the generic element $x = a + b\theta$, $a, b \in \mathbf{C}$, is $||x|| = |a| + |b|$, with $|a|, |b|$ the complex moduli. From this definition, one can show that the norm of the product of any

two elements x and y of the Grassmann algebra satisfies the inequality: $||x \cdot y|| \le ||x|| \cdot ||y||$. This result is true not only for the above simple example but for the general Grassmann algebra generated by $\theta^1, \theta^2, \ldots, \theta^p$. It is also straightforward to show that this norm is complete. In other words, with respect to this norm, the Grassmann algebra becomes a Banach algebra. As we will see below, this allows us to extend to supersymmetric product integrals most of the theorems which apply to ordinary product integrals [12].

Having specified a suitable norm on the Grassmann algebra, we turn to the construction of supersymmetric product integrals and to the study of some of their basic properties.

Definition 1 *Let $\Gamma : [a,b] \to \mathbf{C}^{1|p}_{n\times n}$ be an $n \times n$ matrix valued function with entries in the complex superspace $\mathbf{C}^{1|p}$. Let $P = \{s_0, s_1, \ldots, s_n\}$ be a partition of the interval $[a,b]$, with $\Delta s_k = s_k - s_{k-1}$ for all $k = 1, \ldots, n$.*

(i) Γ is called a step function *iff there is a partition P such that Γ is constant on each open subinterval (s_{k-1}, s_k), for all $k = 1, \ldots, n$.*

(ii) The point value approximant Γ_P *corresponding to the function Γ and partition P is the step function taking the value $\Gamma(s_k)$ on the interval $(s_{k-1}, s_k]$ for all $k = 1, \ldots, n$.*

(iii) If Γ is a step function, then we define the function $E_\Gamma : [a,b] \to \mathbf{C}^{1|p}_{n\times n}$ by $E_\Gamma(x) := e^{\Gamma(s_k)(x-s_{k-1})} \ldots e^{\Gamma(s_2)\Delta s_2} e^{\Gamma(s_1)\Delta s_1}$ for any $x \in (s_{k-1}, s_k]$, for all $k = 1, \ldots, n$, and $E_\Gamma(a) := I$.

Based on the product integral formalism developed for ordinary matrices [12], we want the functions E_Γ to converge to the product integral as the partition of $[a,b]$ is refined. For the proof of the existence of the supersymmetric product integral we need some preliminary results. We start with estimating the norm of E_Γ. This requires one more ingredient, namely the norm of a Grassmann algebra valued matrix. This will be defined in analogy with that of ordinary matrices: for any $n \times n$ matrix Γ as above, we define

$$||\Gamma||_M = \sup_{x \in \mathbf{C}^{n|p} \; ||x||_n \le 1} \frac{||\Gamma\, x||_n}{||x||_n}, \tag{2}$$

where $||x||_n$ refers to the norm of x as an element in $(\mathbf{C}^{1|p})^n \equiv \mathbf{C}^{n|p}$. For $x = (x_1, \ldots, x_n) \in \mathbf{C}^{n|p}$ we can define $||x||_n$ for example by $||x||_n = \sum_{i=1}^n ||x_i||$. Some clarifications are necessary at this point. The Grassmann algebra $\mathbf{C}^{1|p}$ is a $\mathbf{C}$ vector space, but it is not a field. As a result $\mathbf{C}^{n|p}$ is not a vector space over $\mathbf{C}^{1|p}$, but only a rank n module. Though it is a $\mathbf{C}$ vector space, it has no canonical norm on it. With all these preparations we have:

$$\begin{aligned} ||E_\Gamma(x)||_M &= ||e^{\Gamma(s_k)(x-s_{k-1})} \ldots e^{\Gamma(s_2)\Delta s_2} e^{\Gamma(s_1)\Delta s_1}||_M \le \\ &\le ||e^{\Gamma(s_k)(x-s_{k-1})}||_M \ldots ||e^{\Gamma(s_1)\Delta s_1}||_M \le e^{\int_a^x ||\Gamma(s)||_M ds}. \end{aligned} \tag{3}$$

In summary, we have obtained the following result:

$$||E_\Gamma(x)||_M \le e^{\int_a^x ds ||\Gamma(s)||_M}. \tag{4}$$

As a final preparation, we prove the following lemma: Let $\Gamma_1, \Gamma_2 : [a,b] \to \mathbf{C}^{1|p}_{n\times n}$ be step-functions. Then,

$$E_{\Gamma_1}(x) - E_{\Gamma_2}(x) = E_{\Gamma_2}(x) \int_a^x ds\, E^{-1}_{\Gamma_2}(s)[\Gamma_1(s) - \Gamma_2(s)] E_{\Gamma_1}(s). \tag{5}$$

To prove this, we define $G(x) = E^{-1}_{\Gamma_2}(x) E_{\Gamma_1}(x)$. It follows immediately that $G(a) = I$ and $G(x)$ is continuous, and differentiable except for the division points of the partitions associated to Γ_1 and Γ_2. As a result, except for the division points, we have

$$G'(x) = E^{-1}_{\Gamma_2}(x)[\Gamma_1(x) - \Gamma_2(x)] E_{\Gamma_1}(x). \tag{6}$$

The quantity $G(x)$ is continuous and is continuously differentiable on each open division subinterval. Then, using the fundamental theorem of calculus on the subintervals and piecing the results together, we get:

$$G(x) = I + \int_a^x ds\, E^{-1}_{\Gamma_2}(s)[\Gamma_1(s) - \Gamma_2(s)] E_{\Gamma_1}(s). \tag{7}$$

Multiplication from the left by $E_{\Gamma_2}(x)$ leads to Eq. (5).

We are now in a position to define the supersymmetric product integral, and prove its existence:

Definition-Theorem 1 *Given a continuous function $\Gamma : [a,b] \to \mathbf{C}^{1|p}_{n\times n}$ and a sequence of step functions $\{\Gamma_n\}$, which converges to Γ in the $L^1([a,b])$ sense, then the sequence $\{E_{\Gamma_n}(x)\}$ converges uniformly on $[a,b]$ to a matrix called the* supersymmetric product integral *of Γ over $[a,b]$.*

To prove the existence of super product integrals, we must demonstrate the convergence of the sequence $\{E_{\Gamma_n}(x)\}$. By the lemma given above, we have

$$E_{\Gamma_n}(x) - E_{\Gamma_m}(x) = E^{-1}_{\Gamma_m}(x) \int_a^x ds\, E_{\Gamma_m}(s)[\Gamma_n(s) - \Gamma_m(s)] E_{\Gamma_n}(s). \tag{8}$$

We can estimate the norm of the left-hand-side (lhs) as follows:

$$||E_{\Gamma_n}(x) - E_{\Gamma_m}(x)||_M \le ||E^{-1}_{\Gamma_m}(x)||_M \int_a^x ds\, ||E_{\Gamma_m}(s)||_M\, ||\Gamma_n(s) - \Gamma_m(s)||_M\, ||E_{\Gamma_n}(s)||_M. \tag{9}$$

Using Eq. (4), we can estimate the difference of the norms as

$$||E_{\Gamma_n}(x) - E_{\Gamma_m}(x)||_M \le e^{2\int_a^b ds ||\Gamma_m(s)||_M} e^{\int_a^b ds ||\Gamma_n(s)||_M} \int_a^x ds\, ||\Gamma_n(s) - \Gamma_m(s)||_M. \tag{10}$$

Since $\{\Gamma_n\}$ converges to Γ in the $L^1([a,b])$ sense, the first two terms on the rhs are bounded, and the sequence $\{\Gamma_n\}$ is Cauchy in the $L^1([a,b])$ sense. Accordingly, the rhs goes to zero as $m, n \to \infty$. Since the rhs is independent of x, $\{E_{\Gamma_n}\}$ is uniformly Cauchy, hence uniformly convergent. This establishes the existence the supersymmetric product integral. To prove the uniqueness of the limit, we estimate the difference $||E_{B_n}(x) - E_{C_n}(x)||_M$ for two sequences $\{B_n\}$ and $\{C_n\}$, converging to Γ in the L^1 sense. Proceeding as we did above,

it is immediate that $\{E_{B_n}\}$ and $\{E_{C_n}\}$ have the same limit. This concludes the proof of the existence and uniqueness of the supersymmetric product integral.

The structure of the supersymmetric product integrals described above permits the generalization of some of the well-known theorems of product integration [12] to the supersymmetric case. Here we give a summary of the results which are relevant to the proof of the supersymmetric non-Abelian Stokes' theorem. The proofs and further discussion of these results will be given elsewhere [15].

We will follow the notation and the conventions of [11] as much as possible. Let $\Gamma : [a,b] \to \mathbf{C}^{1|p}_{n\times n}$ be a continuous Grassmann valued function. For any $x \in [a,b]$, we express the supersymmetric product integral from a to x as

$$F(x,a) := \prod_a^x e^{\Gamma(s)ds}. \tag{11}$$

Then, F satisfies the integral equation:

$$F(x,a) = 1 + \int_a^x ds\, \Gamma(s) F(s,a). \tag{12}$$

It is also a solution of the initial value problem:

$$\frac{dF}{dx}(x,a) = \Gamma(x)F(x,a), \quad F(a,a) = I. \tag{13}$$

The determinant of a supersymmetric product integral is given by

$$\det\left(\prod_a^x e^{\Gamma(s)ds}\right) = e^{\int_a^x \mathrm{Str}\,\Gamma(\mathrm{s})\mathrm{ds}}, \tag{14}$$

where Str stands for supertrace. The intuitive composition rule holds:

$$\prod_a^b e^{\Gamma(s)ds} = \prod_c^b e^{\Gamma(s)ds} \prod_a^c e^{\Gamma(s)ds}. \tag{15}$$

It is possible to differentiate with respect to the endpoints:

$$\frac{\partial}{\partial x}\left(\prod_y^x e^{\Gamma(s)ds}\right) = \Gamma(x)\prod_y^x e^{\Gamma(s)ds}, \qquad \frac{\partial}{\partial y}\left(\prod_y^x e^{\Gamma(s)ds}\right) = -\prod_y^x e^{\Gamma(s)ds}\Gamma(y). \tag{16}$$

The L-derivative of ordinary product integrals [12] can be extended to super product integrals: for a non-singular differentiable Grassmann valued function $\Gamma : [a,b] \to \mathbf{C}^{1|p}_{n\times n}$, we define

$$L\,\Gamma(x) := \Gamma'(x)\Gamma^{-1}(x), \tag{17}$$

where prime indicates differentiation with respect to x. Defining

$$P(x) = \prod_a^x e^{\Gamma(s)ds}, \tag{18}$$

and using Eq. (13), we can extend the analog of the fundamental theorem of calculus to super product integrals:

$$\prod_a^x e^{(LP)(s)ds} = P(x)P^{-1}(a). \tag{19}$$

The proof of the super non-Abelian Stokes' theorem given below will rely heavily on the contents of the next three theorems. The first one is the *sum rule*. With $P(x) = \prod_a^x e^{\Gamma_1(s)ds}$, we have

$$\prod_a^x e^{[\Gamma_1(s)+\Gamma_2(s)]ds} = P(x) \prod_a^x e^{P^{-1}(s)\Gamma_2(s)P(s)ds}. \tag{20}$$

The second one is the *similarity rule*:

$$P(x)\left(\prod_a^x e^{\Gamma_2(s)ds}\right)P^{-1}(a) = \prod_a^x e^{[LP(s)+P(s)\Gamma_2(s)P^{-1}(s)]ds}. \tag{21}$$

Finally, the third one is *differentiation with respect to a parameter.* Given a Grassmann valued function $\Gamma : [a,b] \times [c,d] \to \mathbf{C}^{1|p}_{n\times n}$ satisfying proper differentiability conditions, and given $P(x,y;\lambda) = \prod_y^x e^{\Gamma(s;\lambda)ds}$, we have:

$$\frac{\partial}{\partial\lambda}P(x,y;\lambda) = \int_y^x ds\, P(x,s;\lambda)\frac{\partial\Gamma}{\partial\lambda}(s;\lambda)P(s,y;\lambda). \tag{22}$$

3 Supersymmetric Wilson Lines and Loops

Our results for supersymmetric product integrals are fairly general. In this section, we will use them as a basis to provide a natural and mathematically sound definition of supersymmetric Wilson lines and loops. To this end, we introduce our notations in a manner which naturally arises in supersymmetric gauge theories. We focus on the supersymmetric Wilson loop first. Consider an oriented manifold M and a closed path C in M. For simplicity, we assume that the target space is a simply connected manifold M, i.e. $\pi_1(M) = 0$. This insures that the loop may be taken to be the boundary of an orientable two dimensional surface Σ in M. It will be convenient to describe the properties of such a 2-surface in terms of local coordinates $\sigma^0 = \tau$ and $\sigma^1 = \sigma$. So, for the points of the manifold M, which lie on Σ, we have $x = x(\sigma,\tau)$.

Let, in standard two component spinor notation [14], the local coordinates of a superspace be given by $z^M = (x^{\alpha\dot{\alpha}}, \theta^\alpha, \theta^{\dot{\alpha}})$. Also let the components of a supersymmetric connection Γ be given by Γ_M. In terms of local coordinates, the connection Γ is a Lie superalgebra valued superform, which can be expressed as $\Gamma = dz^M\Gamma_M$. From the point of view of covariance under supersymmetry transformations, it is more convenient to express Γ in a basis in which the exterior derivative operator $d = dz^M\,\partial_M$ maps superfields to superfields [14]. So, we shall work, instead, in the basis where $d = e^A D_A$, with D_A the supersymmetric covariant derivative, and $e^A(z) = dz^M e_M{}^A(z)$. In this expression, $e_M{}^A(z)$ are the well-known super-beins. Thus, we have

$$\Gamma(z) = dz^M\Gamma_M(z) = e^A(z)\Gamma_A(z). \tag{23}$$

To describe Wilson lines and Wilson loops, we need the pull-back of this quantity on the path C in M, described by an intrinsic parameter s: $x^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}}(s)$, $\theta^{\alpha} = \theta^{\alpha}(s)$, and $\theta^{\dot{\alpha}} = \theta^{\dot{\alpha}}(s)$. In terms of the embedding map $i : C \to M$ we have:

$$\Gamma(s) = i^*\Gamma(z) = \partial_s z^M(s)\,\Gamma_M(z(s)). \tag{24}$$

Similarly, to obtain the pull-back of Γ on the 2-surface, we use the supersymmetric vielbeins:

$$\Gamma_a = v_a^A \Gamma_A; \qquad v_a^M = \partial_a z^M; \quad v_a^A = v_a^M e_M{}^A(z). \tag{25}$$

It is the quantity $\Gamma = \Gamma(s)ds$ or $\Gamma = \Gamma_a d\sigma^a$ that we will identify with the matrix valued functions of the supersymmetric product integral formalism described above. The corresponding pull-backs of the components of the supersymmetric covariant derivative on the line and on the 2- surface are given, respectively, by $\frac{\partial}{\partial s}$ and

$$D_a = v_a^A\, D_A = \frac{\partial}{\partial \sigma^a} = \partial_a. \tag{26}$$

The components of the supersymmetric field strength F_{ab} on the 2-surface can be computed in two different ways. The first method is the obvious pull-back of the target space supersymmetric field strength:

$$F_{ab} = v_a^A v_b^B F_{BA} = v_a^M v_b^N F_{NM}. \tag{27}$$

The second method is to make use of the pulled-back connection Γ_a given above:

$$F_{ab} = \partial_a \Gamma_b - \partial_b \Gamma_a + [\Gamma_a, \Gamma_b]. \tag{28}$$

To show the consistency of the above two expresions, multiply both (27) and (28) with the wedge product of differential forms $\frac{1}{2} d\sigma^a \wedge d\sigma^b$ to get the corresponding field strength two-forms on the two-surface. Then, the consistecy amounts to showing that the two field strength expressions are equal. Since on the two-surface $d\sigma^a v_a^M = dz^M$, Eq. (27) becomes $\frac{1}{2} dz^M dz^N F_{NM}$. Moreover, $d\sigma^a \partial_a = d$ on the two-surface, so that Eq. (28) becomes $d\Gamma - \Gamma^2$. But this expression is equal to the previous one by definition [14].

Consider now the continuous map $\Gamma : [a, b] \to \mathbf{R}^{1|4}_{n\times n}$, where the latter is an n by n matrix valued function, with entries in the superspace $\mathbf{R}^{1|4}$, corresponding to the pull-back on the path C. Then, we define the supersymmetric Wilson line in terms of a super product integral as follows:

$$\mathcal{P} e^{\int_a^b \Gamma(s) ds} \equiv \prod_a^b e^{\Gamma(s) ds}, \tag{29}$$

where $\mathcal{P}$ indicates path ordering as defined by the super product integral on the right-hand-side. Anticipating that we will identify the closed path C over which the Wilson loop is defined with the boundary of a 2-surface, it is convenient to work from the beginning with Wilson lines depending on a parameter. Define $\Gamma_a : [\sigma_0, \sigma_1] \times [\tau_0, \tau_1] \to \mathbf{R}^{1|4}_{n\times n}$, where $[\sigma_0, \sigma_1]$ and $[\tau_0, \tau_1]$ are the range of the local coordinates on the two surface Σ. For later convenience, we also define the following elementary supersymmetric Wilson lines:

$$P(\sigma, \sigma_0; \tau) = \prod_{\sigma_0}^{\sigma} e^{v_1^A \Gamma_A(\sigma';\tau) d\sigma'}, \qquad Q(\sigma; \tau, \tau_0) = \prod_{\tau_0}^{\tau} e^{v_0^A \Gamma_A(\sigma;\tau') d\tau'}. \tag{30}$$

To prove the supersymmetric version of the non-Abelian Stokes theorem, we want to make use of super product integration techniques to express the super Wilson loop operator as an integral over a two dimensional surface bounded by the corresponding loop. For this purpose, we define the super Wilson loop operator as

$$W_s[C] = \mathcal{P}\exp\big(\oint_C \Gamma(\tau)d\tau\big) \equiv e^{\oint_C i^*(dz^M\,\Gamma_M)}. \tag{31}$$

In this expression, as in Eq. (24), i^* denotes the pull-back of the embedding $i : C \to M$. We have written this expression in a notation familiar from the physics literature. It is to be understood, however, that the right-hand-side is to be composed of the super product integrals as given in Eq. (29) above. The expression for the supersymmetric Wilson loop depends on the homotopy class of the loop C in M. We can, therefore, parameterize C in any convenient manner consistent with its homotopy class. In particular, we can break up the closed path into piecewise continuous segments, along which either σ or τ remains constant. The composition rule for super product integrals given by Eq. (15) ensures that this break up of the super Wilson loop into super Wilson lines does not depend on the intermediate points chosen on the closed path. Inspired by the typical paths which are used in the actual computations of of both ordinary and supersymmetric Wilson loops (see e.g. [4, 11]), we break up the super Wilson loop into a product of four super Wilson lines. Using the same notation as in the non-supersymmetric case [11], we write

$$W_s[C] = W_4\,W_3\,W_2\,W_1. \tag{32}$$

In this expression, W_k, $k = 1,..,4$, are super Wilson lines such that $\tau = const.$ along W_1 and W_3, and $\sigma = const.$ along W_2 and W_4. We emphasize that $\sigma = const.$ and $\tau = const.$ are arbitrary curves.

To see the advantage of parameterizing the closed path in this manner, consider the exponent of Eq. (31). Along each segment, only one of the terms is non-vanishing. For example, along the segment $[\sigma_0, \sigma]$, we have $\tau' = \tau_0 = const.$ As a result, we obtain:

$$W_1 = P(\sigma, \sigma_0; \tau_0), \quad W_2 = Q(\sigma; \tau, \tau_0), \quad W_3 = P^{-1}(\sigma, \sigma_0; \tau), \quad W_4 = Q^{-1}(\sigma_0; \tau, \tau_0). \tag{33}$$

Using these expressions, the supersymmetric Wilson loop can be expressed as

$$W_s[C] = Q(\sigma_0; \tau, \tau_0)^{-1} P(\sigma, \sigma_0; \tau)^{-1} Q(\sigma; \tau, \tau_0) P(\sigma, \sigma_0; \tau). \tag{34}$$

For definiteness, in the rest of the paper we will confine ourselves to the case in which the 2-surface, Σ, can be covered by a single coordinate patch. If Σ requires more than one patch to be covered, then using partition of unity and the product integral composition rule, Eq. (15), it is straightforward to extend our upcoming reasonings.

4 Super Non-Abelian Stokes Theorem

As an application of the supersymmetric product integral formalism, we prove the supersymmetric version of the non-Abelian Stokes theorem [16]. The proof makes essential use of the generalized theorems listed in the previous paragraphs, and is the supersymmetric

version of one of the proofs given for the non-supersymmetric case in reference [11]. The other proof give in this reference can also be extended to the supersymmetric case and will be given in a subsequent work [15].

We start with the form of $W_s[C]$ given in Eq. (34) and take its derivatives with respect to the parameter τ:

$$\begin{aligned}\frac{\partial W_s[C]}{\partial \tau} \;=\;& \partial_\tau Q^{-1}(\sigma_0;\tau,\tau_0)P^{-1}(\sigma,\sigma_0;\tau)Q(\sigma;\tau,\tau_0)P(\sigma,\sigma_0;\tau_0)+\\ &+Q^{-1}(\sigma_0;\tau,\tau_0)\partial_\tau P^{-1}(\sigma,\sigma_0;\tau)Q(\sigma;\tau,\tau_0)P(\sigma,\sigma_0;\tau_0)+\\ &+Q^{-1}(\sigma_0;\tau,\tau_0)P^{-1}(\sigma,\sigma_0;\tau)\partial_\tau Q(\sigma;\tau,\tau_0)P(\sigma,\sigma_0;\tau_0).\end{aligned} \tag{35}$$

Here, we have made use of the fact that $P(\sigma,\sigma_0;\tau_0)$ is independent of τ. As a preparation for using Eq. (19), we start with Eq. (17) for $W_s[C]$, and make use of Eq. (16) to get

$$\begin{aligned}L_\tau W_s[C]=\frac{\partial W_s[C]}{\partial \tau}W_s[C]^{-1} =\;& T^{-1}(\sigma;\tau)\left[\Gamma_0(\sigma;\tau)-P(\sigma,\sigma_0;\tau)\Gamma_0(\sigma_0;\tau)P^{-1}(\sigma,\sigma_0;\tau)-\right.\\ &\left.-\partial_\tau P(\sigma,\sigma_0;\tau)P^{-1}(\sigma,\sigma_0;\tau)\right]T(\sigma;\tau),\end{aligned} \tag{36}$$

In this expression, $T(\sigma;\tau)=P(\sigma,\sigma_0;\tau)\,Q(\sigma_0;\tau,\tau_0)$. Next, by means of differentiation with respect to a parameter given by Eq. (22), we evaluate the derivative of the super product integral $P(\sigma,\sigma_0;\tau)$ with respect to the parameter τ:

$$\partial_\tau P(\sigma,\sigma_0;\tau)=\int_{\sigma_0}^{\sigma} d\sigma' P(\sigma,\sigma';\tau)\partial_\tau\Gamma_1(\sigma';\tau)P(\sigma',\sigma_0;\tau). \tag{37}$$

Then, after some simple manipulations using the defining equations for the various terms in Eq. (36), we get:

$$T^{-1}(\sigma;\tau)\partial_\tau P(\tau)P^{-1}(\tau)T(\sigma;\tau)=\int_{\sigma_0}^{\sigma} d\sigma' T^{-1}(\sigma';\tau)\partial_\tau\Gamma_1(\sigma';\tau)T(\sigma';\tau). \tag{38}$$

Using Eq. (16) and the fact that $P(\sigma_0,\sigma_0;\tau)=1$, we can rewrite the rest of Eq. (36) also as an integral:

$$\begin{aligned}&T^{-1}(\sigma;\tau)[\Gamma_0(\sigma;\tau)-P(\sigma,\sigma_0;\tau)\Gamma_0(\sigma_0;\tau)P^{-1}(\sigma,\sigma_0;\tau)]T(\sigma;\tau)=\\ &=\textstyle\int_{\sigma_0}^{\sigma} d\sigma'\,P^{-1}(\sigma',\sigma_0;\tau)(\partial_\tau\Gamma_0(\sigma',\tau)+[\Gamma_0(\sigma',\tau),\Gamma_1(\sigma',\tau)])P(\sigma',\sigma_0;\tau).\end{aligned} \tag{39}$$

Combining Eqs. (36), (38), and (39), we obtain:

$$L_\tau W_s[C]=\int_{\sigma_0}^{\sigma} d\sigma' T^{-1}(\sigma',\tau)F_{01}(\sigma',\tau)T(\sigma',\tau). \tag{40}$$

Here F_{01} is the field strength component as defined in Eq. (28), but based on the discussion in that paragraph, we know that it also equals the pull-back of the supersymmetric field strength to the surface. Using Eq. (19), we are immediately led to the supersymmetric version of the non-Abelian Stokes theorem:

$$W_s[C]=\prod_{\tau_0}^{\tau} e^{\int_{\sigma_0}^{\sigma} T^{-1}(\sigma';\tau')F_{01}(\sigma';\tau')T(\sigma';\tau')d\sigma' d\tau'}. \tag{41}$$

Recalling the antisymmetry of the components of the field strength, we can rewrite this expression in a more familiar reparameterization invariant form

$$W_s[C] = \mathcal{P}_\tau e^{\oint \Gamma} = \prod_{\tau_0}^{\tau} e^{\frac{1}{2}\int_\Sigma d\sigma^{ab}\, T^{-1}(\sigma;\tau) F_{ab}(\sigma;\tau) T(\sigma;\tau)}, \tag{42}$$

where $d\sigma^{ab}$ is the area element of the 2-surface. Despite appearances, it must be remembered that σ and τ play very different roles in this expression.

The above result also applies to the special case in which the gauge group is Abelian. In that case, however, since the corresponding matrices commute, the machinery of the super product integrals is not needed, and one can establish the super Stokes theorem directly [17].

5 Gauge Covariance of the Super Loop Operator

To demonstrate the gauge covariance of the supersymmetric Wilson loop operator and its 2-surface representation, we must show how the supersymmetric Wilson line transforms under gauge transformations. For this, we need to know, in turn, how the pull-back of the connection $\Gamma_A(z)$ transforms. The transformation properties of the connection itself follows from that of the vector superfield [14]: $e^{V'} = e^{-i\Lambda^\dagger} e^V e^\Lambda$. More specifically, we have

$$\Gamma'(z) = g(z)\Gamma(z)g(z)^{-1} - g(z)dg(z)^{-1}, \tag{43}$$

where, $g(z) = e^{i\Lambda(z)}$ and $d = e^A(z)D_A$. As we have seen, the pull-back of this quantity on the line is given by $d = ds\partial_s$. Thus, we get for the transformation of the supersymmetric connection on the line:

$$\Gamma'(s) = g(s)\Gamma(s)g^{-1}(s) - g(s)\partial_s g^{-1}(s). \tag{44}$$

This is formally identical to that for the plain Yang-Mills theory [11]. As a result, under a gauge transformation we obtain:

$$\prod_a^b e^{ds\,\Gamma(s)} \longrightarrow \prod_a^b e^{[g(s)\Gamma(s)g^{-1}(s) - g(s)\partial_s g^{-1}(s)]\,ds}. \tag{45}$$

By Eq. (17), we have $g(s)\partial_s g^{-1}(s) = -L_s g(s)$. Thus, for the gauge transformed super Wilson line we have

$$\prod_a^b e^{[g(s)\Gamma(s)g^{-1}(s) + L_s g(s)]\,ds}. \tag{46}$$

Moreover, using Eq. (21) and recalling from Eq. (19) that $\prod_a^b e^{L_s g(s)\,ds} = g(b)g^{-1}(a)$, the gauge transformed expression takes the form

$$g(b)g^{-1}(a) \prod_a^b e^{g(a)\Gamma(s)g^{-1}(a)}. \tag{47}$$

Finally, using the same argument as in reference [11], the constant terms in the exponents can be factored out from the super product integral. Thus, we get for the gauge transformed super Wilson line

$$\prod_a^b e^{ds\,\Gamma(s)} \longrightarrow g(b)\left(\prod_a^b e^{ds\,\Gamma(s)}\right) g^{-1}(a). \tag{48}$$

We can use this result to determine the gauge transforms of operators which are products of simple super Wilson lines. Consider, e.g., the operator $T(\sigma;\tau)$ which is the product of two super Wilson lines. Applying Eq. (48) to each factor, we obtain:

$$T(\sigma;\tau) \longrightarrow g(\sigma;\tau)T(\sigma;\tau)g^{-1}(\sigma_0;\tau_0). \tag{49}$$

From this, we can easily obtain the transformation properties of the super Wilson loop operator which is also a composite of super Wilson lines. The transformation has the same form as Eq. (48) with $a = b$.

Finally, let us consider how the surface integral representation of super Wilson loop operator given by Eq. (41) transforms under gauge transformation. From knowing how each factor in the exponent transforms, it follows that

$$W_s[C] \longrightarrow \prod_{\tau_0}^{\tau} e^{g(\sigma_0;\tau_0)\left(\int_{\sigma_0}^{\sigma} T^{-1}(\sigma';\tau')F_{01}(\sigma';\tau')T(\sigma';\tau')dt'\right)g^{-1}(\sigma_0;\tau_0)}. \tag{50}$$

Just as for super Wilson line, the constant terms in the exponent factorize, so that under gauge transformations the surface integral representation of the super Wilson loop transforms covariantly:

$$W_s[C] \longrightarrow g(\sigma_0;\tau_0)\prod_{\tau_0}^{\tau} e^{\int_{\sigma_0}^{\sigma} T^{-1}(\sigma';\tau')F_{01}(\sigma';\tau')T(\sigma';\tau')dt'} g^{-1}(\sigma_0;\tau_0). \tag{51}$$

6 Concluding Remarks

In this work, we have presented a supersymmetric generalization of ordinary product integral formalism. Given that Wilson lines and Wilson loops can be expressed in terms of ordinary product integrals, we have constructed the supersymmetric extensions of these notions for supersymmetric gauge theories in terms of supersymmetric product integrals. These constructions are natural in the sense that the supersymmetric representations given in this paper reduce to the ordinary product integral representations of standard Wilson lines and Wilson loops.

It is hoped that this formalism provides a reliable non-perturbative means of extracting information from supersymmetric gauge theories. In this respect, we note that the construction of the supersymmetric Wilson lines and Wilson loops as well as the proof of the super non-Abelian Stokes theorem given in the previous sections are independent of any specific physical applications. To apply these concepts to supersymmetric gauge theories, it is necessary to clarify the physical content of the operators such as the connection and the field strength which appear in the relevant expressions [16, 15]. It is well known that in supersymmetric gauge theories the superfield strength F_{AB} contains more degrees of freedom

than is required by supersymmetry and gauge invariance [18]. As a result, it is necessary to impose constraints on the components of the field strength to eliminate the unphysical degrees of freedom. This means that in the expressions for supersymmetric Wilson lines and loops, Γ and F must be expressed in terms of unconstrained superfields, just as in the abelian case [17]. Such a description in terms of unconstrained superfields already exist in the literature [8, 14, 18] and can be adapted to specific applications.

This work was supported in part by the Department of Energy under the contract number DOE-FGO2-84ER40153. We are grateful to M. Awada for valuable input at the initial stages of this work. We would also like to thank R. Grimm, A. Kiss and R.L. Mkrtchian for helpful communications.

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